YDK 517.9

A PLANE STEADY VORTEX FREE BOUNDARY PROBLEM: EXISTENCE AND UNIQUENESS RESULTS USING THE COMPLEX METHOD

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1. Introduction.

The aim of the present paper is the study of a free boundary problem connected with the steady plane irrotational vortex motion for an incompressible fluid. Let us now describe formally this problem (a precise formulation will be done in the next section).

Let us assume that Ω is the region of the complex plane occupied by the fluid and that $0 \in \Omega$ is the singularity of the vortex. If $\Psi : \overline{\Omega} - \{0\} \to R$ is the stream function defining the motion, then we have that Ψ is a harmonic function in $\Omega - \{0\}$ (since the mouvement is steady and irrotational). Near the singularity of the vortex we have also the following asymptotical behaviour: $\Psi(z) \sim -\log|z|$. The abowe facts can be precised by the following relations:

$$\Psi(z) = B(z) - \log|z| \text{ in } \overline{\Omega} - \{0\}, \tag{1.1}$$

where:

$$B : \overline{\Omega} \to R \text{ such that } \Delta B = 0 \text{ in } \Omega. \tag{1.2}$$

We assume also that Ω is "like a ball" which wraps around the singularity of the vortex in 0. Two conditions apply on the (free) boundary of Ω (c_1, c_2, g being real constants, the constant g being the gravity acceleration):

$$\Psi = c_1, \quad \frac{1}{2}|(\nabla\Psi)(z)|^2 + g \operatorname{Im} z = c_2, \quad \text{on } \partial\Omega. \tag{1.3}$$

The first one of such relations tell us that the boundary of Ω is a streamline; the second one is a consequence of Bernoulli law (assuming the external pressure as a constant). Notice that in (1.3), the constant c_1 is arbitrary, whereas the constant c_2 is an unkown quantity. To obtain a well-posed problem, we must impose a further condition: for instance we can specify the value of the speed in a assigned point of Ω (see Problem A later).

In what follows the previous problem is studied using a complex method. I begin giving a precise complex mathematical formulation of the problem (see Problem A in section 2) and, in section 3, (by a suitable analytical transformation) reducing it to a new problem defined in the unit ball of the complex plane. The solution of the (transformed) problem can be represented by an integral formula using the classical complex Poisson kernel (see section 4). This fact allows us to characterize the boundary value of the (transformed) solution as a fixed point for a suitable non-linear operator (see Problem B in section 5). This formulation can be considered as a weak formulation of the physical problem since different solutions of the original problem give rise to different solutions of the weak formulation. Afterwords (section 6) an existence theorem is given for the weak formulation (for any value of $g \geq 0$) and an uniqueness theorem is also given (when $g \geq 0$ is small enough). These results imply also an uniqueness theorem of the physical formulation given by Problem A (always when $g \geq 0$ is small enough).

2. Precise mathematical formulation of the problem.

Since Ω is "like a ball", to the harmonic function B introduced in section 1, we can associate a harmonic function $A : \overline{\Omega} \to R$ (with $A(0) = 0$) such that $A + iB$

is a holomorphic function on Ω. As usual we could consider the speed potential function:

$$\Phi(z) = A(z) + \arg z/i, \tag{2.1}$$

where arg is the principial branch of the argument function. Then we have that the function $\Phi + i\Psi$ is a holomorphic function only in the open set:

$$\Omega' = \Omega - \{z \in \Omega : \ \operatorname{Im} z \leq 0, \ \ \operatorname{Re} z = 0\}.$$

More suitable is the use of the holomorphic function $F : \overline{\Omega} \to C$ (given up to a real constant) given by $A + i B$. Recalling (1.1), we obtain that $\Psi(z) = \operatorname{Im} F(z) - \log|z|$. Hence:

$$\frac{\partial \Psi}{\partial y}(z) + i\frac{\partial \Psi}{\partial x}(z) = F'(z) - \frac{i}{z}, \quad z \in \overline{\Omega} - \{0\}. \tag{2.2}$$

In terms of the function F, the problem described 1 can then be stated precisely in the following way:

Problem A. Given $g \geq 0$, we look for a couple $\{\Omega, F\}$ where Ω is an open subset of C with $0 \in \Omega$ and such that there exists an analytical, simple, closed, positively oriented path $\gamma : [-1,1] \to C$ having Ω as the unique bounded connected component of $C - \gamma([-1,1])$ and:

$$\overline{\gamma(t)} = -\gamma(-t), \quad t \in [-1,1], \tag{2.3}$$

$$\operatorname{Im}\gamma(1) \leq \operatorname{Im}\gamma(t) \leq \operatorname{Im}\gamma(0), \quad t \in [-1,1] \tag{2.4}$$

Moreover $F \in H(\overline{\Omega})$ (that is F is a holomorphic function in an open set containing $\overline{\Omega}$) with $\operatorname{Re} F(0) = 0$. On the boundary of Ω we prescribe also:

$$Im\, F(z) = \log|z|, \quad |F'(z) - i/z|^2 + 2g \operatorname{Im} z = \text{constant}, \quad z \in \partial\Omega, \tag{2.5}$$

$$F'(\gamma(0)) = \frac{1 - |\gamma(0)|}{|\gamma(0)|}. \tag{2.6}$$

By the hyphotheses it turns out that γ (the boundary of Ω) is a closed path which is symmetric with respect to the imaginary axis and which winds, in the positive sense, exactly one time, around the points of Ω. We also have that $\gamma(0)$ (resp. $\gamma(1)$) is the top (resp. the bottom) of $\gamma([-1,1])$ (and of $\overline{\Omega}$). The relation (2.5) is the translation of (1.3) in terms of the function F (with $c_1 = 0$). By the relation (2.2), we can verify easily that the relation (2.6) simply prescribes that the speed in the highest point of $\gamma([-1,1])$ is 1. Notice that the constant in (2.5) is not a datum of the problem.

If $g = 0$ then $F \equiv 0$ and Ω given by the unit disc of the complex plane is a solution of Problem A.

3. Transformation of the problem.

From now on, $\{\Omega, F\}$ will be a solution of Problem A. We can now construct the following holomorphic function:

$$\Lambda : \overline{\Omega} \to C \ \text{ defined by } \ \Lambda(z) = -iz\exp(iF(z)). \tag{3.1}$$

Remark 3.1. It easy to verify that the function Λ can be expressed in the following way (the function Φ is defined in (2.1)):

$$\Lambda(z) = \exp[i(\Phi(z) + i\Psi(z))].$$

where this relation has a meaning.

We can now prove ($D(0,1)$ being the unit disc of C):

Proposition 3.2. The map Λ is a conformal mapping of $\overline{\Omega}$ on $\overline{D(0,1)}$. We have also:

$$\Lambda(-\overline{z}) = \overline{\Lambda(z)}, \quad \Lambda'(-\overline{z}) = -\overline{\Lambda'(z)}, \quad z \in \overline{\Omega}, \tag{3.2}$$

$$\Lambda(\gamma(0)) = 1, \quad \Lambda(\gamma(-1)) = \Lambda(\gamma(1)) = -1, \quad \Lambda'(\gamma(0)) = -i, \tag{3.3}$$

$$\Lambda'(z) = z\,[F'(z) - \frac{i}{z}]\exp(i\,F(z)), \quad z \in \overline{\Omega} - \{0\}, \tag{3.4}$$

$$|\Lambda'(z)|^2 + 2g\,\mathrm{Im}\,z = \text{constant}, \quad |\Lambda'(z)| \geq 1, \quad z \in \partial\Omega. \tag{3.5}$$

Proof: By the maximum principle it follows that

$$\mathrm{Im}\,F(z) > \log|z|, \quad z \in \Omega - \{\emptyset\}. \tag{3.6}$$

By the definition of Λ, it follows that $|\Lambda(z)| = |z|\exp(-\mathrm{Im}\,F(z))$, $(z \in \overline{\Omega})$. Hence, recalling (2.5) and (3.6), we obtain:

$$\Lambda(\partial\Omega) \subset \partial D(0,1), \quad \Lambda(\Omega) \subset D(0,1). \tag{3.7}$$

If we put $\Gamma(t) = \Lambda(\gamma(t))$, $t \in [-1,1]$, the we have that Γ is a smooth path and that $\Gamma^* = \Gamma([-1,1]) \subset \partial D(0,1)$. By the argument principle it follows that:

$$n(\Gamma,0) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}dz = \frac{1}{2\pi i}\int_\gamma \frac{\Lambda'(z)}{\Lambda(z)}\,dz = N,$$

where $n(\Gamma,0)$ is the winding number of Γ around 0 and N is the number of zeros of Λ in Ω. (counted according to the multiplicity). By (3.1) we have that $\Lambda(z) = 0$ if and only if $z = 0$ (this zero being simple): hence $n(\Gamma,0) = 1$. Taking into account of (3.7), it follows that $\Gamma^* = \partial D(0,1)$. Since $n(\Gamma,\omega)$ is constant in every connected component of $C - \Gamma^*$, we obtain that $n(\Gamma,\omega) = 1$, for all $\omega \in D(0,1)$, that is:

$$1 = n(\Gamma,\omega) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-\omega}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{\Lambda'(z)}{\Lambda(z)-\omega}\,dz = M, \quad \omega \in D(0,1),$$

where M is the number of zeros of the function $z \to \Lambda(z) - \omega$ in Ω. Hence for all $\omega \in D(0,1)$ there exists one and only one $z \in \Omega$ such that $\Lambda(z) = \omega$. This fact proves that Λ is a conformal mapping of Ω on $D(0,1)$. Using well known results (see, for instance, [1], cap. VI, Theorem 4) we obtain soon that Λ can be extended analytically to the boundary.

The relations (3.2)-(3.4) can be obtained easily using the definition of Λ and the simmetry of Ω. The relations (3.5) are a consequence of (2.5).

Thanks to Proposition 3.2, we can introduce the following function:

$$L : \overline{D(0,1)} \to C \quad \text{defined by:} \quad L(\omega) = i\,\Lambda'(\Lambda^{-1}(\omega)) = \frac{i}{(\Lambda^{-1})'(\omega)}. \tag{3.8}$$

Proposition 3.3. We have that:

$$L \in H(\overline{D(0,1)}), \tag{3.9}$$

$$\overline{L(\overline{\omega})} = L(\omega) \neq 0, \quad \omega \in \overline{D(0,1)}, \tag{3.10}$$

$$L(1) = 1, \tag{3.11}$$

$$1 \leq |L(e^{iv})|^4 = 1 + 4g\int_0^v \mathrm{Im}\,[e^{it}\,\overline{L(e^{it})}]\,dt, \quad v \in [-\pi,\pi]. \tag{3.12}$$

Proof: The relations (3.9)-(3.10) are easy consequences of the simmetry of Ω and of Proposition 3.2. Taking $z = \Lambda^{-1}(e^{iv})$ in (3.5), we have:

$$|\Lambda'(\Lambda^{-1}(e^{iv}))|^2 + 2g \operatorname{Im} \Lambda^{-1}(e^{iv}) = \text{constant}, \quad v \in [-\pi, \pi]. \tag{3.13}$$

By an easy calculation it follows that:

$$\frac{d}{dv}(\Lambda^{-1}(e^{iv})) = -\frac{\exp(iv)}{L(\exp(iv))} = -\frac{\exp(iv)\overline{L(\exp(iv))}}{|L(\exp(iv))|^2}$$

and differentiating the relation (3.13), we obtain:

$$\frac{d}{dv}|\Lambda'(\Lambda^{-1}(e^{iv}))|^2 = 2g\,\frac{\operatorname{Im}[\exp(iv)\overline{L(\exp(iv))}]}{|L(\exp(iv))|^2}$$

which implies the equality in (3.12) (the inequality in (3.12) being a consequence of (3.5)). Finally the relation (3.11) can be obtained by an easy calculation.

4. Integral representation of the problem.

Let us now consider the classical complex Poisson kernel:

$$H_r(v) = \frac{1 + re^{iv}}{1 - re^{iv}}, \quad r \in [0, 1], \quad v \in R.$$

Put also:

$$P_r(v) = \operatorname{Re} H_r(v), \quad Q_r(v) = \operatorname{Im} H_r(v), \tag{4.1}$$

which are (respectively) the ordinary (real) Poisson kernel and the so-called conjugate Poisson kernel. Then we have:

Theorem 4.1. *The function L may be represented as:*

$$L(re^{iv}) = exp(\frac{1}{8\pi}\int_{-\pi}^{\pi} H_r(v-t)\log|L(e^{it})|^4\,dt), \quad (r, v) \in [0, 1[\times[-\pi, \pi]. \tag{4.2}$$

Proof: By (3.9) and (3.10) we obtain that $\log|L|$ is a harmonic function in $\overline{D(0,1)}$. Hence, by the mean value theorem, it follows that:

$$\log|L(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|L(e^{iv})|\,dv = \frac{1}{8\pi}\int_{-\pi}^{\pi}\log|L(e^{iv})|^4\,dv. \tag{4.3}$$

Put now:

$$G(re^{iv}) = \exp(\frac{1}{8\pi}\int_{-\pi}^{\pi} H_r(v-t)\log|L(e^{it})|^4\,dt), \quad (r, v) \in [0, 1[\times[-\pi, \pi].$$

We must prove that $L \equiv G$ in $D(\emptyset, 1)$. Notice that:

$$\lim_{r\uparrow 1}|G(re^{iv})| = \lim_{r\uparrow 1}\exp(\frac{1}{8\pi}\int_{-\pi}^{\pi} P_r(v-t)\log|L(e^{it})|^4\,dt) = |L(e^{iv})|,$$

and then, by the maximum modulus theorem for holomorphic functions, this implies that $|G(z)/L(z)| \le 1$ $(z \in D(0,1))$. Since, by (4.3), we have that $G(0) = |L(0)|$, it follows that $|G(z)| = |L(z)|$ $(z \in D(0,1))$. Because $\lim_{r\uparrow 1} = |L(1)| = L(1)$, the theorem is proved.

Theorem 4.1 gives an integral representation of the function L. We remark also that the relation (4.2), connected with the equality (3.12), suggests a fixed point procedure to characterize the boundary value of L (that is the values of $L(e^{iv})$, with $v \in [-\pi, \pi]$). Actually, if we propose a starting value of L on $\partial D(0,1)$, we can evaluate the value of $|L(e^{iv})|^4$ (using the relation (3.12)). Replacing after this value in the integral representation formula (4.2), we obtain the value of L in $D(0,1)$. Finally, taking the trace of L on $\partial D(0,1)$, we must find again the starting value of L on the boundary. This fixed point procedure will be used to study the present problem (see later). This method can be also employed for the numerical treatment of the present problem (see [6]).

5. Reduction to the boundary.

If we put $E(v) = L(e^{iv})$, taking $r \uparrow 1$ in (4.2), it follows (recalling also (4.1)):

$$E(v) = |E(v)| \lim_{r\uparrow 1} \exp\left(\frac{i}{8\pi} \int_{-\pi}^{\pi} Q_r(v-t) \log |E(t)|^4 \, dt\right), \quad v \in R. \tag{5.1}$$

It is well known that (see for instance [5]):

$$\lim_{r\uparrow 1} \left(\frac{i}{8\pi} \int_{-\pi}^{\pi} Q_r(v-t)\, h(t)\, dt = (\Xi\, h, v), \quad v \in R, \quad h \in C^{0,1}(R,R), \tag{5.2}$$

where Ξ is (a variant of) the so-called conjugate operator defined by:

$$(\Xi\, h)(v) = (\Xi\, h, v) = -\frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{h(v+t) - h(v-t)}{2\mathrm{tg}(t/2)}\, dt.$$

Notice that the definition of Ξ, suitably adapted by the use of a principalvalue integral, can be extended to $h \in L^1$ (see again [5]). Roughly speaking, the mathematical meaning of the conjugate operator is the following: to the trace on the boundary of a harmonic function the conjugate operator associates the trace of its harmonic conjugate function.

Let now ($\alpha \in]0,1]$):

$$W_\alpha = \{h \in C^{0,\alpha}(R,R) : \ h(v) = h(-v) = h(v+2\pi), \ v \in R\}, \tag{5.3}$$

with the norm:

$$\|h\|_\alpha = \sup\{|h(v)|, \ v \in]-\pi,\pi[\} + \sup\left\{\frac{|h(v+t) - h(v)|}{|t|^\alpha}, \ v,t \in]-\pi,\pi[\text{with } t \neq 0\right\}.$$

We have that ($\alpha \in]0,1[$) :

$$\Xi : \ W_\alpha \to W_\alpha \ \text{ is a linear and continuous map,} \tag{5.4}$$

which is a result due to Fatou [3] (see also [2] or [4]). Set now:

$$\chi = \{\eta \in C^0(R,C) : \ \eta(v) = \overline{\eta(-v)} = \eta(v+2\pi), \ v \in R\}, \tag{5.5}$$

with the norm:

$$\|\eta\|_0 = \sup\{|\eta(v)|, \ \ v \in]-\pi,\pi[\}$$

and define:

$$\mathcal{M} = \{\eta \in \chi : \ \|\eta\|_0 < \exp(g\pi)\}.$$

Put now:

$$P(x) = \begin{cases} x, & \text{if } x \geq 1, \\ (x^2+1)/2, & \text{if } x \in [0,1[, \\ \frac{1}{2}, & \text{if } x < 0. \end{cases}$$

and $(\lambda \geq 0,\ \ \eta \in \chi)$:

$$(S_\lambda\, \eta)(v) = (S_\lambda\, \eta; v)_{\text{def}} = P(1 + 4\lambda \int_0^v \operatorname{Im}[e^{it}\overline{\eta(t)}]\, dt), \quad v \in R. \tag{5.6}$$

Thanks to (3.12), we have that:

$$|E(v)|^4 = (S_g\, E)(v), \quad v \in R. \tag{5.7}$$

Let us consider the map:

$$\mathcal{F} :\ \overline{\mathcal{M}} \times [0, g] \to \chi,$$

defined by:

$$(\mathcal{F}_\lambda\, \eta)(v) = (S_\lambda\, \eta)^{1/4}\, (v) \exp(i(\Xi \log S_\lambda\, \eta)(v)).$$

We can now state the following:

Problem B. We look for a function $E \in \overline{\mathcal{M}}$ such that $\mathcal{F}_g\, E = E$.

We have that:

Theorem 5.1. *The function $E(v) = L(e^{iv})$ is a solution of Problem B.*

Proof: By Gronwall Lemma and (5.7), we obtain easily that $|E(v)| \leq \exp{(g\pi)}$ $(v \in R)$: hence $E \in \overline{\mathcal{M}}$. Recalling the relations (5.1), (5.2) and (5.7), finally we obtain that $E = \mathcal{F}_g\, E$.

Remark 5.2. It is immediate to verify that two different solutions of Problem A give raise to different solutions of Problem B. Hence Problem B can be interpreted as a weak formulation of Problem A.

6. Existence and uniqueness results for Problem B.

By an easy calculation (for more details, see [7]), it follows the following result in the case in which the gravity acceleration is small enough:

Theorem 6.1. *There exists $\delta > 0$ sufficiently small such that, if $0 \leq g \leq \delta$, then $\mathcal{F}(\overline{\mathcal{M}}) \subset \overline{\mathcal{M}}$ and the map $\mathcal{F}\ :\ \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ is a contraction mapping. Hence, if g is small enough Problem B has one and only one solution.*

Remark 6.2. By the previous theorem and by Remark 5.2, we obtain an uniqueness result for Problem A in the case in which g is small enough.

The following further existence result for Problem B can be obtained by the method of topological degree:

Theorem 6.3. *For every $g \geq 0$ there exists a solution of Problem B.*

Proof: If $g = 0$, $\eta \equiv 1$ is the only solution of Problem B.

Let us assume now $g > 0$. An easy application of Gronwall Lemma gives us:

$$\forall \eta \in \mathcal{M}, \quad \forall\, \lambda \in [0, g]\ (\mathcal{F}_\lambda\, \eta = \eta \Rightarrow \eta \in \mathcal{M}), \tag{6.1}$$

which means that no fixed point of $\mathcal{F}_\lambda (\lambda \in [0, g[)$ exists in $\partial\mathcal{M}$.

Since $\mathcal{F}_{0\eta} \equiv 1 \in \mathcal{M}$, it follows that (see [8], Th. 4.3.1 and Th. 4.3.6):

$$\deg{(I - \mathcal{F}_0, \mathcal{M}, 0)} = \deg(I - 1, \mathcal{M}, 0) = \deg(I, \mathcal{M}, 1) = 1. \tag{6.2}$$

Let also:

$$Z :\ \overline{\mathcal{M}} \times [0, g] \to W_1,$$

defined by:

$$(Z_\lambda\, \eta)(v) = (\log S_\lambda\, \eta)(v),$$

It is easy to verify that the map Z is continuous. The use of (5.4) yields:

$$\Xi :\ W_1 \to \chi \ \text{ is a linear and completely continuous map.} \tag{6.3}$$

This result implies that $\mathcal{F}$ is a homotopy of compact transformations on $\mathcal{M}$; hence (see [8] Th. 4.3.4):

$$\deg{(I - \mathcal{F}_0, \mathcal{M}, 0)} = \deg{(I - \mathcal{F}_g, \mathcal{M}, 0)} = 1, \tag{6.4}$$

which says that there exists $E \in \mathcal{M}$ such that $E - \mathcal{F}_g E = 0$.

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